

# LOCAL ATTRACTOR CONTINUATION OF NON-AUTONOMOUSLY PERTURBED SYSTEMS

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**ABSTRACT.** Using Conley theory we show that local attractors remain (past) attractors under small non-autonomous perturbations. In particular, the attractors of the perturbed systems will have positive invariant neighborhoods and converge upper semicontinuously to the original attractor.

The result is split into a finite-dimensional part (locally compact) and an infinite-dimensional part (not necessarily locally compact). The finite-dimensional part will be applicable to bounded random noise, i.e. continuous time random dynamical systems on a locally compact metric space which are uniformly close the unperturbed deterministic system. The “closeness” will be defined via a (simpler version of) convergence coming from singular perturbations theory.

## 1. INTRODUCTION

This paper uses methods from Conley index theory to establish a continuation of isolated attractors. For these attractors there is a stable neighborhood, i.e. a positive invariant isolating neighborhood. Traditionally the Conley index is applied to isolated invariant set whose isolating neighborhoods are bounded and satisfy some compactness assumption, Rybakowski [Ryb87] calls them admissible neighborhoods. Focusing on a stable neighborhood we show that a non-autonomously perturbed system has also a stable neighborhood which is “close” to the original one.

A similar result for parabolic PDEs was obtained by Prizzi in [Pri05] for small almost-periodic perturbations with sufficiently high “frequency”. Because almost-periodicity causes compactness of the “perturbation-space” he can obtain non-empty invariant sets even for the unstable case (for flows on locally compact spaces our result also applies to totally unstable invariant sets). In [War94] Ward obtains the result for ODEs and small perturbations satisfying some hypothesis (H1). Our result generalizes both of them in a way that we don’t need almost-periodicity and applies to quite general semiflows even in the infinite-dimensional setting.

An open question to us is if an unstable invariant set with non-trivial Conley index disappears for all small non-autonomous perturbations. The continuation result and the Ważewski principle only give us a non-empty positive invariant set.

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Obviously this cannot happen in dimension 1 and the perturbation should neither be almost-periodic nor satisfy Ward's hypothesis (H1).

Furthermore, our result implies the upper semicontinuity of the global pullback attractor (see [CL03]) if the perturbation is "uniformly small". In addition, this also holds for local attractors (called past attractors in [Ras07]).

The result (see section 3) is a consequence of the translation invariance of the unperturbed flow and standard continuation results of the Conley index. The proof is essentially contained in [Ryb87, Theorem 12.3] after replacing all admissibility arguments by an appropriate version (see also [CR02]).

For locally compact metric spaces  $X$  we can use ideas from [Ben91]. Because the proof is very clear and easy to understand, we are going to show it in section 2 even though this case, which we call finite-dimensional case, is contained in the infinite-dimensional case. Another reason to give the proof is that the obtained stable neighborhoods are flow-defined and thus applicable to random dynamical system on locally compact metric spaces without further assumptions (see section 2.2). This implies that sufficiently small bounded noise does not destroy a local attractor. We don't need regularity of the support of the noise used by Ruelle in [Rue81] and thus we could generalize his result (see also [BDV05, Appendix D] and the reference therein).

Furthermore, using the ideas of [Kel11b] and adjusting the definition of semi-singular admissibility the infinite-dimensional case can be extended to the discrete time setting. Thus stable neighborhoods of a local attractor of a discrete time dynamical system, i.e. a continuous maps  $f : X \rightarrow X$ , can be continued under small non-autonomous perturbations  $\tilde{f} : \mathbb{Z} \times X \rightarrow \mathbb{Z} \times X$ .

### Preliminaries.

**Definition 1** (local semiflow). Let  $\pi : D \rightarrow X$  be a continuous map into a topological space  $X$  and  $D$  open in  $\mathbb{R}^+ \times X$  with  $\{0\} \times X \subset D$ . For all  $x \in X$  we define

$$\omega_x = \sup\{t > 0 \mid (t, x) \in D\} \in (0, \infty].$$

Assume  $D \cap \mathbb{R} \times \{x\} = [0, \omega_x) \times \{x\}$ . Then  $\pi$  is called a (local) semiflow if the following holds

- $x\pi 0 = x$  for all  $x \in X$
- if  $(t, x) \in D$  and  $(s, x\pi t) \in D$  then  $x\pi(s+t)$  is defined and equals  $(x\pi t)\pi s$

If, in addition,  $\omega_x = \infty$  for all  $x \in X$  then  $\pi$  is called a global semiflow.

*Remark.* We use the notation  $x\pi t$  for  $\pi(t, p)$  and  $x\phi^p t = \phi(t, p, x)$  whenever  $(t, p, x) \in D$  (see below). Furthermore, we write  $x\phi^p[0, t_0]$  for the set  $\{x\phi^p t \mid t \in [0, t_0]\}$  under the condition that  $t_0 < \omega_x^p$ , otherwise  $x\phi^p[0, t_0]$  is not defined.

**Definition 2** (non-autonomous dynamical system (NDS)). Let  $X$  be a metric space and  $P$  be a set called the base set. A (local) NDS is a pair of mappings

$$\begin{aligned} \theta : \mathbb{R} \times P &\rightarrow P \\ \phi : D &\rightarrow X \end{aligned}$$

such that the following holds:

- $\theta$  is a (not necessarily continuous) flow, called the base flow

- $D$  is open in  $\mathbb{R}^{\geq 0} \times P \times X$  with  $\{0\} \times P \times X \subset D$ ,  $x\phi^p 0 = x$  and whenever  $(s, p, x) \in D$  and  $(s + t, p, x) \in D$  for some  $s, t \geq 0$

$$(t, p\theta s, x\phi^p s) \in D$$

and

$$x\phi^p(t + s) = (x\phi^p s)\phi^{p\theta s} t.$$

Furthermore, we define

$$\omega_x^p = \sup\{t > 0 \mid (t, p, x) \in D\}$$

- $\phi$  is continuous with respect to  $t \in \mathbb{R}_{\geq 0}$  and  $x \in X$  for fixed  $p \in P$ .

We call  $(\phi, \theta)$  a (local) semiproduct if  $\theta$  is continuous and  $P$  a metric space.

*Remark.* To every (semi)process  $(\phi, \theta)$  we can associated a semiflow  $\pi : D \rightarrow P \times X$  in the following way. If  $(t, p, x) \in D$  then  $(p, x)\pi t = (p\theta t, x\phi^p t)$ . The resulting (semi)flow is called skew product semiflow.

*Remark.* If  $P$  is a compact metric space and  $\theta$  continuous then standard Conley index theory is applicable: The unperturbed system  $\pi$  is a product of the flow  $\theta$  and a (semi)flow  $\pi_0$  on  $X$  so that a Conley index  $h(N, \pi_0)$  lifts to

$$h(P \times N, \theta \times \pi_0) = h(P, \theta) \wedge h(N, \pi_0).$$

The stability result follows from a standard continuation result for the index and  $H_0(h(P \times N, \theta \times \pi_0)) \neq 0$  iff  $H_0(h(N, \pi_0)) \neq 0$  (see [Kel11a] for classification of stability via the zeroth singular homology  $H_0$  of the index).

Although most of the following can be formulated for general NDS if we assume uniform convergence w.r.t. to the orbit set of  $\theta$  (see section 2.1) we will restrict our attention to  $P = \mathbb{R}$  and  $\tau_t(s) := \theta(t, s) = t + s$ . This includes processes generated by non-autonomous differential equations. Furthermore, we will only look at the induced skew product (semi)flow, i.e.  $\pi : D \subset \mathbb{R}^{\geq 0} \times \tilde{X} \rightarrow \tilde{X}$  with  $\tilde{X} = \mathbb{R} \times X$ .

A map  $\sigma : J \rightarrow X$  with  $J \subset \mathbb{R}$  is called a solution of  $\pi$  through  $x_0 \in X$  if  $0 \in J$  and  $\sigma(0) = x_0$  and whenever  $t, t + s \in J$  for some  $s > 0$  with  $s < \omega_{\sigma(t)}$  then

$$\sigma(t + s) = \sigma(t)\pi s.$$

$\sigma$  is a left solution if  $J \cap \mathbb{R}^- = (a, 0]$  for some  $a \in [-\infty, 0)$  and it is called a full left solution if  $\mathbb{R}^- \subset J$ .

Let  $Y \subset X$  be arbitrary. We define the following sets

$$A^+(Y) = \{x \in Y \mid x\pi t \in Y \text{ for all } t \in [0, \omega_x)\}$$

$$A^-(Y) = \{x \in Y \mid \exists a \text{ full left solution } \sigma \text{ through } x \text{ with } \sigma(\mathbb{R}^-) \subset Y\}$$

and

$$A(Y) = A^-(Y) \cap A^+(Y).$$

$Y$  is called invariant if  $Y = A(Y)$ , positive invariant if  $Y = A^+(Y)$  and negative invariant if  $Y = A^-(Y)$ . The sets  $A(Y)$  and  $A^\pm(Y)$  depend on  $Y$  and  $\pi$ . In case we talk about several flows  $\pi_n$  we will write  $A_{\pi_n}(Y)$  and  $A_{\pi_n}^\pm(Y)$ .

If  $S \subset \text{int } N$  for some closed neighborhood  $N$  and  $S$  is the maximal invariant set in  $N$ , i.e.  $S = A(N)$ , then  $S$  is called an isolated invariant set (w.r.t.  $\pi$ ). A closed set  $N$  is called isolating neighborhood if the maximal  $\pi$ -invariant set is in the interior of  $N$ , i.e.  $A(N) \subset \text{int } N$ . In particular the closure  $\text{cl } U$  of a neighborhood  $U$  of  $A(N)$  with  $U \subset N$  is an isolating neighborhood.

**Definition 3.** A closed isolating neighborhood  $N$  is called stable if

$$A(N) \subset \text{int } N$$

and  $N$  is positive invariant, i.e.

$$A^+(N) = N.$$

## 2. FINITE DIMENSIONAL CASE

Although many ideas in this section are taken from Conley index theory we don't want to introduce the full theory of the Conley index. Many of the techniques and definitions in this section are based on Benci's paper [Ben91]. From now on let  $X$  be locally compact. Since our isolating neighborhood will be compact we will assume that  $\pi_0$  is a flow in order to simplify our arguments, i.e. there will be no finite-time blow-up. This will be true for the skew product flow induced by a process if the “ $X$ -component” of set  $\tilde{N} \subset \mathbb{R} \times X$  is compact as well (the unbounded component represents time).

From [Ben91] we take the following definitions:

$$G^T(N) = G_\pi^T(N) = \{x \in X \mid x\pi[-T, T] \subset \text{cl } N\}$$

and

$$\Gamma^T(N) = \Gamma_\pi^T(N) = \{x \in G^T(N) \mid x\pi[0, T] \cap \partial N \neq \emptyset\}.$$

Furthermore we define the set of isolating neighborhoods as

$$\mathcal{F} = \mathcal{F}_\pi = \{N \subset X \mid \text{int } N \neq \emptyset \text{ and } \exists T > 0 \text{ s.t. } G^T(N) \subset \text{int } N\}.$$

**Lemma 4** ([Ben91]). *If  $N \in \mathcal{F}$  then the following hold:*

- (1)  $T_1 > T_2 > 0$  then  $G^{T_1}(N) \subset G^{T_2}(N)$
- (2)  $G^T(N)$  and  $\Gamma^T(N)$  are closed and  $\Gamma^T(N) \subset \partial G^T(N)$
- (3) If  $G^T(N) \subset \text{int } N$  then  $G^{2T}(N) \subset \text{int } G^T(N)$
- (4) If  $\Gamma^T(N) = \emptyset$  then  $A^+(G^T(N)) = G^T(N)$

*Remark.* It can be shown that for large  $T$  the pair  $(G^T(N), \Gamma^T(N))$  defines an index pair (see section 3 for the definition). We will focus only on stable neighborhoods for which  $\Gamma^T(N) = \emptyset$  for large  $T$ . We can show that for arbitrary isolated invariant set with non-trivial Conley index the index continues even for sufficiently small non-autonomous perturbations. But the corresponding index pair  $(N_1, N_2)$  is unbounded. In particular, we cannot show in general that  $A(N_1) \neq 0$  if the index  $(N_1/N_2, [N_2])$  is non-trivial.

A standard result from Conley index theory is the following theorem. This result also holds under the assumption that  $N$  is strongly  $\pi$ -admissible (see section 3 for the definition).

**Theorem 5** ([Ryb87, Corollary 5.5]). *Let  $N$  be a compact isolating neighborhood such that*

$$A^-(N) = A(N) = K \neq \emptyset$$

*then there exist a stable isolating neighborhood  $B \subset \text{int } N$  such that for all  $x \in \partial B$  there is an  $\epsilon > 0$  such that for all  $t \in (0, \epsilon)$*

$$x\pi(-t) \notin B$$

*and*

$$x\pi t \in \text{int } B.$$

An isolating neighborhood with this property will be called *stable isolating block*.

*Remark.* Suppose we got  $B' \subset N$  as a result from the theorem. Applying the theorem again we get a second stable isolating block  $B \subset B'$ . Because  $B'$  is compact and  $K$  is in its interior,  $U_\delta(B) = \{x \in X \mid d(x, B) < \delta\} \subset B'$  for some  $\delta > 0$ .

Let  $\pi_0$  be a flow on  $X$ . Then we can define a skew product flow  $\pi$  on  $\mathbb{R} \times X$  by

$$(s, x)\pi t = (\tau_s(t), x\pi_0 t).$$

Thus  $\pi_0$  can be considered as a process which does not depend on the base flow  $\tau$  and is therefore translation invariant w.r.t. time.

**Definition 6** (Semi-singular convergence). Let  $\pi_n$  be a sequence of semiflows on  $\mathbb{R} \times X$  and  $(s_n, x_n)_{n \in \mathbb{N}}$  be any sequence in  $\mathbb{R} \times X$  and  $t_n \in \mathbb{R}^{\geq 0}$  such that  $x_n \rightarrow x_0$  and  $t_n \rightarrow t_0$  then we say  $\pi_n$  converges semi-singularly to  $\pi_0$  if

$$P_2((s_n, x_n)\pi_n t_n) \rightarrow x_0\pi_0 t_0$$

where  $P_2(s, x) = x$ . We write  $\pi_n \xrightarrow{\text{ssing}} \pi_0$ . Semi-singular convergence implies that  $\pi_n \rightarrow \pi$  in the usual sense if each  $\pi_n$  is skew product flow of processes because if in addition  $s_n \rightarrow s_0$  for some  $s_0 \in \mathbb{R}$  then

$$(s_n, x_n)\pi_n t_n = (s_n + t_n, P_2((s_n, x_n)\pi_n t_n)) \rightarrow (s_0 + t_0, x_0\pi_0 t_0) = (s_0, x_0)\pi t_0.$$

*Remark.* Semi-singular convergence is a simplified version of singular convergence defined in [CR02]. With the metric  $d_\epsilon(t, s) = \epsilon \min\{|t-s|, 1\}$  we recover the singular convergence.

Assume from now on that  $\pi_n$  is a sequence of flows such that  $\pi_n \xrightarrow{\text{ssing}} \pi_0$ .

**Example.** The standard example is an autonomous ordinary differential equation

$$\dot{x} = f_0(x)$$

and the non-autonomous ODEs

$$\dot{x} = f_n(t, x)$$

such that  $\sup_t \|f_n(t, x) - f_0(x)\| < \epsilon_n$  with  $\epsilon_n \rightarrow 0$ . If  $f_0$  and  $f_n$  are locally Lipschitz continuous then they induce a local flow  $\pi_0$ , resp. local processes  $\phi_n$ . Restricted to some compact set in  $X$  we can assume that  $\pi_0$  and  $\phi_n$  are defined everywhere. Suppose we have  $(s_n, x_n)_{n \in \mathbb{N}}$  with  $x_n \rightarrow x_0$  and  $t_n \rightarrow t_0$ . The ODEs defined by

$$\dot{x} = \tilde{f}_n(t, x) = f_n(s_n + t, x)$$

generate time-translated processes  $\tilde{\phi}_n$ . We still have  $\sup_t \|\tilde{f}_n(t, x) - f_0(x)\| < \epsilon_n$ . The sequence  $(s_n, x_n)_{n \in \mathbb{N}}$  corresponds to  $(0, x_n)_{n \in \mathbb{N}}$ . Interpreting  $\phi_n$  and  $\tilde{\phi}_n$  as skew products  $\pi_n$  and  $\tilde{\pi}_n$  we see that  $\tilde{\pi}_n \rightarrow \pi = \tau \times \pi_0$  which implies semi-singular convergence of  $\pi_n$ .

Suppose now  $S \subset X$  is an attractor for  $\pi_0$ . Then there exists a stable isolating block  $B$ . Thus  $N := \mathbb{R} \times B$  is a stable isolating block for  $\pi$  with  $A_\pi(N) = \mathbb{R} \times S$ . Furthermore, it can be shown that for all  $T > 0$

$$\begin{aligned} G_\pi^T(N) &= \mathbb{R} \times G_{\pi_0}^T(B) \\ &= \mathbb{R} \times (B\pi_0 T) \end{aligned}$$

and  $\Gamma_\pi^T(N) = \emptyset$ .

Since  $B$  is compact  $(\cdot)\pi t : N \rightarrow N$  is uniformly continuous for all  $t \geq 0$ . And for each  $T > 0$  there is some  $\delta > 0$  such that  $U_\delta(G_\pi^T(N)) \subset N$  where  $U_\epsilon(A) = \{x \in X \mid d(x, A) < \epsilon\}$ .

We assume here that  $\pi_n$  does not blow-up in finite time in  $N$ . This will always be the case if  $\pi_n$  is a skew product flow because the unbounded component of  $N = \mathbb{R} \times B$  represents the time and  $B$  is compact. Hence whenever  $\omega_{(s,x)}^{\pi_n} < \infty$  for  $(s, x) \in N$  then  $(s, x)\pi_n[0, \omega_{(s,x)}^{\pi_n}) \not\subset N$ .

**Lemma 7.** *For every  $T > 0$  there is an  $n_0 > 0$  such that for all  $n \geq n_0$*

$$G_{\pi_n}^T(N) \neq \emptyset.$$

*Proof.* easy exercise □

**Lemma 8.** *There is a  $T_0$  and a  $n_0$  such that for all  $T \geq T_0$  and  $n \geq n_0$*

$$\Gamma_{\pi_n}^T(N) = \emptyset.$$

*Remark.* The proof is based on [CR02, Lemma 3.5]. The idea is to show that if it does not hold then  $K = A_{\pi_0}(B)$  and  $\partial B$  intersect non-trivial which is impossible because  $K$  is isolated. We will give the whole proof because the fact that  $\tilde{z} \in A_{\pi_0}^-(B)$  will also apply to the infinite-dimensional case if we assume  $\{\pi_n\}$ -semi-singular-admissibility of  $N$ .

*Proof.* Suppose this is not the case. Then there exist a sequence  $T_n \rightarrow \infty$  and a subsequence of  $\pi_n$ , also denoted by  $\pi_n$ , such that  $E_n := \Gamma_{\pi_n}^{T_n}(N) \neq \emptyset$  for all  $n$ .

This implies that there is a sequence  $x_n \in N$  such that for some  $s_n \in [0, T_n]$  we have  $x_n\pi_n[0, t_n] \subset N$ ,  $x_n\pi_n T_n \in E_n$  and  $x_n\pi_n t_n \in \partial N$  with  $t_n = T_n + s_n$ .

Since  $N = \mathbb{R} \times B$  we have  $\partial N = \mathbb{R} \times \partial B$ . So that  $x_n\pi_n t_n = (r_n, \tilde{z}_n)$  for some  $\tilde{z}_n \in \partial B$ . Because  $\partial B$  is compact there is a subsequence of  $x_n\pi_n t_n$  denoted by  $z_n^0 = x_n^0\pi_n^0 t_n^0$  such that  $\tilde{z}_n^0 \rightarrow \tilde{z}^0 \in \partial B$ .

Since  $B$  is compact,  $t_n^0 \rightarrow \infty$  and  $x_n^0\pi_n^0[0, t_n^0] \subset N$  there is a subsequence  $(x_n^1\pi_n^1 t_n^1)_{n \in \mathbb{N}}$  with  $t_n^1 \geq 1$  such that  $(r_n^1, \tilde{z}_n^1) = x_n^1\pi_n^1(t_n^1 - 1)$  is defined and  $\tilde{z}_n^1 \rightarrow \tilde{z}^1 \in B$ . Recursively, for each  $k \geq 1$  we get a subsequence  $(x_n^k\pi_n^k t_n^k)_{n \in \mathbb{N}}$  of  $(x_n^{k-1}\pi_n^{k-1} t_n^{k-1})_{n \in \mathbb{N}}$  with  $t_n^k \geq k$  such that  $z_n^k = (r_n^k, \tilde{z}_n^k) = x_n^k\pi_n^k(t_n^k - k)$  and  $\tilde{z}_n^k \rightarrow \tilde{z}^k$  for some  $\tilde{z}^k \in B$ .

We claim that  $\tilde{z}^k\pi_0 t \in N$  all  $t \in [0, k]$  and  $\tilde{z}^k\pi_0 k = \tilde{z}^0$ , i.e.  $\tilde{z}^0 \in A_{\pi_0}^-(B)$ . Postponing the proof of this claim we immediately get that  $\tilde{z}^0 \in \partial B \cap A^-(B)$  but  $B$  is a stable isolating neighborhood such that  $A(B) = A^-(B) \subset \text{int } B$ , i.e.  $\partial B \cap A_{\pi_0}^-(B) = \emptyset$ , which is a contradiction and thus the lemma is true.

It remains to proof our claim. Suppose  $\tilde{z}^k\pi_0 t_0 \notin B$  for some  $t_0 \in [0, k]$ . Because  $\pi_n \xrightarrow{\text{ssing}} \pi_0$  the sequence  $\tilde{y}_n$  with  $(s_n, \tilde{y}_n) = z_n^k\pi_n^k t_0$  converges to  $\tilde{z}^k\pi_0 t_0$ , but  $\tilde{y}_n \in B$  and  $B$  is closed which is impossible. Hence  $\tilde{z}^k\pi_0[0, k] \subset B$ . □

Combining the two previous lemmas we obtain that there is a  $T_0$  and an  $n_0$  such that for all  $T \geq T_0$  and  $n \geq n_0$  the set  $G_{\pi_n}^T(N) \neq \emptyset$  is a stable isolating neighborhood for  $\pi_n$ .

Because  $B$  is compact and  $(\cdot)\pi t$  is uniformly continuous on  $N$ , there is a  $\delta = \delta(\pi_0, B) > 0$  such that for  $T \geq T_0$

$$U_\delta(G_\pi^{2T}(N)) \subset G_\pi^T(N)$$

and

$$U_\delta(G_\pi^T(N)) \subset N.$$

Furthermore, choose  $\delta$  small enough and  $T$  large enough so that  $U_\delta(B) \subset B'$  (see remark after theorem 5) and

$$G_{\pi_0}^T(B') \subset B,$$

i.e.  $U_\delta(G_\pi^T(U_\delta(N))) \subset N$  holds as well.

*Remark.* Benci ([Ben91]) defines a set  $\Sigma_0$  which contains all sets having a similar property and uses it to prove the continuation of the Conley index. Most proofs of the inclusions like  $U_\delta(G_\pi^{2T}(N)) \subset G_{\pi_n}^T(N)$  are omitted in that paper.

In addition assume now that each  $\pi_n$  is a skew product flow, i.e.

$$P_1((s, x)\pi_n t) = s + t.$$

**Theorem 9.** *If for large  $n \geq n_0$  and  $T \geq T_0$  we have*

$$d(x\pi_n t, x\pi t) < \frac{\delta}{3}$$

*for all  $t \in [-T, T]$  and  $x \in N$  then*

$$A_{\pi_n}(N) \neq \emptyset$$

*is an isolated invariant set with compact  $t$ -slices such that for an  $\epsilon = \epsilon(\pi_0, B, T) > 0$*

$$U_\epsilon(A_{\pi_n}(N)) \subset G_{\pi_n}^T(N).$$

*Remark.* The theorem shows that  $A_{\pi_n}(N)$  is a past attractor in the sense of Rasmussen [Ras07].

*Proof.* Let  $x \in G_{\pi_n}^T(N)$  then

$$x\pi_n[-T, T] \subset N.$$

Since  $d(x\pi t, x\pi_n t) < \delta$  for all  $t \in [-T, T]$  we have

$$x\pi[-T, T] \subset U_\delta(N).$$

This implies that  $G_{\pi_n}^T(N) \subset G_\pi^T(U_\delta(N))$  and thus  $U_\delta(G_{\pi_n}^T(N)) \subset N$  because  $U_\delta(G_\pi^T(U_\delta(N))) \subset N$ .

Because  $\pi$  is uniformly continuous on  $N \times [-T, T]$  we can choose  $\epsilon > 0$  such that  $d(x, \tilde{x}) < \epsilon$  implies  $d(x\pi t, \tilde{x}\pi t) < \frac{\delta}{3}$  for all  $x, \tilde{x} \in N$  and  $t \in [-T, T]$ . Now if  $x \in U_\epsilon(G_{\pi_n}^{2T}(N))$  then there is an  $\tilde{x} \in G_{\pi_n}^{2T}(N)$  such that  $d(x, \tilde{x}) < \epsilon$  and

$$\tilde{x}\pi_n[-T, T] \subset G_{\pi_n}^T(N).$$

Furthermore, we have for all  $t \in [-T, T]$

$$\begin{aligned} d(x\pi_n t, \tilde{x}\pi_n t) &\leq d(x\pi_n t, x\pi t) + d(x\pi t, \tilde{x}\pi t) + d(\tilde{x}\pi t, \tilde{x}\pi_n t) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned}$$

Since  $U_\delta(G_{\pi_n}^T(N)) \subset N$  we have  $x\pi_n[-T, T] \subset N$ , i.e.  $U_\epsilon(G_{\pi_n}^{2T}(N)) \subset G_{\pi_n}^T(N)$ . In particular,  $U_\epsilon(A_{\pi_n}(N)) \subset G_{\pi_n}^T(N)$ .

Now let  $x \in G_\pi^{2T}(N)$  then  $x\pi[-T, T] \subset G_\pi^T(N)$ . Because  $U_\delta(G_{\pi_n}^T(N)) \subset N$  and  $d(x\pi t, x\pi_n t) < \delta$  for all  $t \in [-T, T]$  we have  $x \in G_{\pi_n}^T(N)$ . Hence

$$G_\pi^{2T}(N) = \mathbb{R} \times G^{2T}(B) \subset G_{\pi_n}^T(N).$$

Since  $G_n := G_{\pi_n}^T(N)$  is closed the set  $G_n(t) = G_{\pi_n}^T(N) \cap \{t\} \times X \subset \{t\} \times B$  is compact and non-empty for each  $t \in \mathbb{R}$ . Now define

$$A(t) = \bigcap_{t \geq 0} G_n(-t) \pi_n t \subset \{t\} \times B.$$

$G_n$  is positive invariant so that this intersection is a decreasing sequence of non-empty compact sets and thus compact and non-empty itself. Let

$$A = \bigcup_{t \in \mathbb{R}} A(t).$$

Then  $A$  is invariant w.r.t.  $N$  and  $A(N) \subset A \subset N$  which implies  $A(N) = A \neq \emptyset$ .  $\square$

**2.1. Non-autonomous dynamical systems.** For a general NDS  $(\phi, \theta)$  we do not have  $P = \mathbb{R}$ . Assume  $X$  is locally compact and  $(\phi, \theta)$  is a two-sided process. Denote by  $P^*$  the set of  $\theta$ -orbits of  $P$ . Then each orbit  $\sigma \in P^*$  is represented by a  $p \in P$ . So after choosing one representative for each orbit we can assume w.l.o.g.  $P^* \subset P$ .

Because  $(\phi, \theta)$  is two-sided for each  $p \in P^*$  we have an invertible skew product flow  $\pi_p$  with

$$(s, x) \pi_p t = (s + t, x \phi^{p\theta s} t).$$

Suppose  $B$  is a stable isolating neighborhood of  $\pi_0$ . If each  $\pi_p$  is close to  $\pi = \tau \times \pi_0$ , i.e. there is a sufficiently small  $\delta > 0$  and a  $T > 0$  (both independent of  $p$ ) such that

$$d((s, x) \pi t, (s, x) \pi_p t) < \delta$$

for all  $t \in [-T, T]$  and  $x \in B$  then theorem 9 implies

$$G_p = G^T(\mathbb{R} \times B)$$

is a stable isolating neighborhood for  $\pi_p$  and their isolated invariant sets  $A_p = A_{\pi_p}(\mathbb{R} \times B)$  are non-empty. Now define the set-valued map  $D : P \rightarrow 2^X$  by

$$D(q) = \{x \in X \mid \text{for some } s \in \mathbb{R}, p \in P^* \text{ s.t. } (s, x) \in G_p \text{ and } p\theta s = q\}$$

It is an easy exercise to show that  $D$  maps into the set of closed sets and is forward invariant w.r.t. the non-autonomous system  $(\phi, \theta)$ . In the same way we can show that there is set-valued map  $A$  which is invariant and

$$A(q) = P_2((A_p \pi_p s) \cap \{s\} \times X)$$

is compact for  $q = p\theta s$  and  $p \in P^*$ . Furthermore, the  $\epsilon$  in theorem 9 only depends on  $T$ ,  $B$  and  $\pi_0$ , i.e.

$$U_\epsilon(A(q)) \subset D(q)$$

which shows that  $A$  is a past attractor.

**2.2. Random dynamical systems.** A random dynamical system is an NDS such that  $P = \Omega$  is a probability space,  $\theta$  is measurable dynamical system and the map

$$(t, \omega) \mapsto x \phi^\omega t$$

is measurable for every  $x$  in  $X$ .

If we show that the sets  $D$  and  $A$  are random closed sets then  $A$  is a random past attractor. And we get:

**Theorem.** *Sufficiently small (bounded) noise does not destroy local attractors.*



In particular, if the global attractor of the unperturbed system has several disjoint local attractors (“sinks”) then the same holds for the global attractor of the perturbed system. This is a complementary result to Crauel, Flandoli - “Additive Noise Destroys a Pitchfork Bifurcation” [CF98] where they show that the “perturbed” global attractor is a single point with probability 1 when “small” white noise is applied.

It remains to show that  $D$  and  $A$  are closed random sets. It suffices to show that  $D$  is random compact. Each  $G_\omega^T(N)$  is flow-defined, i.e.

$$G_\omega^T(N) = \bigcap_{t \in [-T, T]} (\mathbb{R} \times B) \pi_\omega t.$$

Because the  $\pi_\omega$  are skew products, for each  $s$ -slice we have

$$\begin{aligned} G_\omega^T(N)(s) &= G_\omega^T(N) \cap (\{s\} \times X) \\ &= \{s\} \times \bigcap_{t \in [-T, T]} (B\phi^{\omega\theta(s-t)}t). \end{aligned}$$

So that for  $\varpi = \omega\theta s$

$$\begin{aligned} D(\varpi) &= \bigcap_{t \in [-T, T]} (B\phi^{\varpi\theta(-t)}t) \\ &= \bigcap_{t \in [-T, T] \cap \mathbb{Q}} (B\phi^{\varpi\theta(-t)}t) \end{aligned}$$

Because  $B$  is a deterministic compact set,  $\phi$  invertible and  $D$  a decreasing intersection of compact sets,  $D$  is a random compact set. In particular, we see that the choice of the representative  $\omega \in \Omega^* = P^*$  does not matter and each  $D(\varpi)$  is defined in the same way.

### 3. INFINITE DIMENSIONAL CASE

In this section we show that the index pair continuation in [Ryb87, Theorem 12.3] still holds if we replace the admissibility arguments by semi-admissibility and semi-singular convergence arguments. We are not able to show that two index pairs of the perturbed (non-autonomous) system are Conley equivalent, i.e. the quotients have the same homotopy type in the category of pointed space. So the Conley index is not well-defined. Even worse, because the index pair is unbounded the Ważewski principle does not imply that the invariant set is non-empty if the quotient space is non-trivial. Nevertheless, we can use the continuation to show that a stable isolating neighborhood continues to a stable isolating neighborhood, i.e. the exit set of the perturbed system is empty.

*Remark.* As mentioned in the introduction appropriate versions of theorem 14 and corollary 15 hold also for stable isolating neighborhoods in the discrete time setting using the ideas of [Kel11b]. The definitions of semi-(singular-)admissibility and semi-singular convergence can be given similarly.

Furthermore, for both, continuous and discrete time, instead of a semi-admissibility of the perturbed maps we could use “weak properness” of the space  $X$ , i.e. bounded  $\delta$ -neighborhoods of compact sets are weakly compact, to show that the invariant set is non-empty and a weak attractor (see [Kel11b] for definition).

**Definition 10** (Semi-singular admissibility). Let  $\pi_n$  be semiflows on  $\mathbb{R} \times X$  and  $\pi_0$  be a semiflow on  $X$  such that  $\pi_n \xrightarrow{\text{ssing}} \pi_0$ . A closed subset  $N$  of  $\mathbb{R} \times X$  is called  $\{\pi_n\}$ -semi-singular-admissible (ss-admissible) if the following holds:

- Let  $(s_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . If  $x_n \in X$  and  $t_n \in \mathbb{R}^+$  with  $0 \leq t_n < \omega_{(s_n, x_n)}^{\pi_n}$  are two sequences such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $(s_n, x_n)\pi_n[0, t_n] \subset N$  then the sequence  $\{P_2((s_n, x_n)\pi_n t_n)\}_{n \in \mathbb{N}}$  of the projected endpoints has a convergent subsequence.

If, in addition, each  $\pi_n$  does not explode in  $N$ , i.e.

$$(s, x)\pi_n[0, \omega_{(s, x)}^{\pi_n}) \subset N \text{ implies } \omega_{(s, x)}^{\pi_n} = \infty$$

then we say that  $N$  is strongly  $\{\pi_n\}$ -ss-admissible.

The definition means that  $\pi_n$  should not converge too badly to  $\pi_0$ . It is a property of the sequence  $\{\pi_n\}_{n \in \mathbb{N}}$  and does not tell anything about  $\pi_0$  alone and neither about a single  $\pi_n$ . An adjusted admissibility definition of a single semiflow  $\pi$  for our setting is given as follows.

**Definition 11** (Semi-Admissibility). Let  $\pi$  be a semiflow on  $\mathbb{R} \times X$ . A closed subset  $N$  of  $\mathbb{R} \times X$  is called  $\pi$ -semi-admissible if the following holds:

- Let  $(s_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . If  $x_n \in X$  and  $t_n \in \mathbb{R}^+$  with  $0 \leq t_n < \omega_{(s_n, x_n)}^{\pi}$  are two sequences such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\{s_n + t_n\}_{n \in \mathbb{N}}$  is precompact,  $(s_n, x_n)\pi[0, t_n] \subset N$  then the sequence of endpoints  $\{(s_n, x_n)\pi t_n\}_{n \in \mathbb{N}}$  has a convergent subsequence.

If, in addition,  $\pi$  does not explode in  $N$ , i.e.

$$(s, x)\pi[0, \omega_{(s, x)}) \subset N \text{ implies } \omega_{(s, x)} = \infty$$

then we say that  $N$  is strongly  $\pi$ -semi-admissible.

*Remark.* For a semiflow  $\pi_0$  on  $X$  the usual strong  $\pi_0$ -admissibility of a closed subset  $N$  is defined without the sequence  $s_n$  (see [Ryb87, 4.1]).

Suppose  $B$  is closed and strongly  $\pi_0$ -admissible,  $B \subset U$  for some open  $U \subset X$  with strong  $\pi_0$ -admissible closure and  $d(B, \partial U) \geq \delta$ . We say that  $\pi_n \xrightarrow{\text{ssing}} \pi_0$  converges uniformly if  $d(P_2((s, x))\pi_n t^*, x\pi_0 t^*) < \epsilon_{n, t} \rightarrow 0$  for  $t > 0$  and all  $(s, x) \in \mathbb{R} \times U$  with  $t < \min\{\omega_x^{\pi_0}, \omega_{(s, x)}^{\pi_n}\}$ .

*Remark.* Uniform convergence of  $\pi_n \xrightarrow{\text{ssing}} \pi_0$  is similar to the assumption of Benci [Ben91] used to prove his continuation theorem for the Conley index.

**Lemma 12.** Suppose  $\pi_n \xrightarrow{\text{ssing}} \pi_0$  uniformly and  $\pi_n$  does not explode in  $N = \mathbb{R} \times B$ . Then  $N$  is strongly  $\{\pi_n\}$ -admissible.

*Proof.* Let  $(s_n, x_n) \in N$  and  $t_n \rightarrow \infty$  be a sequence fulfilling the assumption of the definition of ss-admissibility. Because of uniform convergence and  $d(N, \partial U) \geq \delta$ , there is an  $N(t)$  for each such that for all  $n \geq N(t)$

$$x\pi_0[0, t] \subset U$$

whenever  $(s, x) \in N$  with  $(s, x)\pi_n t \subset N$ .

Thus there is a sequence  $r_n \geq 0$  with  $t_n - r_n \rightarrow \infty$  such that  $(s_n, x_n)\pi_n[0, t_n] \subset N$  implies that

$$y_n\pi_0[0, t_n - r_n] \subset U$$

for  $(\tilde{s}_n, y_n) = (s_n, x_n)\pi_n r_n$ . Furthermore, we can choose  $r_n$  such that  $\delta_n = \max_{t \in [0, t_n - r_n]} \epsilon_{n,t} \rightarrow 0$  and therefore

$$d(P_2((s_n, x_n)\pi_n t_n), y_n \pi(t_n - r_n)) \leq \delta_n.$$

Because the closure of  $U$  is strongly  $\pi_0$ -admissible, the sequence of endpoints  $\{y_n \pi(t_n - r_n)\}_{n \in \mathbb{N}}$  has a convergent subsequence which implies that  $\{P_2((s_n, x_n)\pi_n)\}_{n \in \mathbb{N}}$  has a convergent subsequence, in particular the limit point is in  $N$ .  $\square$

**Definition 13** (index pair). Let  $\tilde{N}$  be a closed isolating neighborhood for  $K = A(\tilde{N})$ . A pair  $(N, L)$  is called an index pair in  $\tilde{N}$  (w.r.t.  $\pi$ ) if  $L \subset N \subset \tilde{N}$  and the following holds

- (1)  $K \subset \text{int}(N \setminus L)$  and  $K$  is the maximal invariant set in  $\text{cl}(N \setminus L)$
- (2) if  $x \in L$  and  $x\pi[0, \epsilon] \subset N$  then  $x\pi[0, \epsilon] \subset L$ , i.e.  $L$  is positive  $\pi$ -invariant relative to  $N$
- (3) if  $x \in N$  and  $x\pi[0, \omega_x] \not\subset N$  then there is a  $0 \leq t < \omega_x$  such that  $x\pi t \in L$ , i.e.  $L$  is an exit ramp for  $N$

The following theorem is a variant of [Ryb87, I-12.3-12.7] (see also [CR02]) by replacing all admissibility arguments with semi-singular-admissibility argument and replacing  $g^- : \tilde{N} \rightarrow \mathbb{R}^{\geq 0}$  by a lifted version  $\tilde{g}^- : \mathbb{R} \times \tilde{N} \rightarrow \mathbb{R}^{\geq 0}$  defined by  $\tilde{g}^-(s, x) = g^-(x)$ .

**Theorem 14.** Let  $\pi_0$  be a local semiflow on  $X$  and  $\pi_n$ ,  $n \in \mathbb{N}$ , be local semiflows on  $\mathbb{R} \times X$ . Suppose  $\tilde{N}$  is a closed set in  $X$  and strongly  $\pi_0$ -admissible. Moreover, assume  $\pi_n \xrightarrow{\text{sing}} \pi_0$  as  $n \rightarrow \infty$  and  $\mathbb{R} \times \tilde{N}$  is strongly  $\{\pi_{n_m}\}$ -semi-admissible for all subsequences  $\{\pi_{n_m}\}_{m \in \mathbb{N}}$ . Set  $K_n = A_{\pi_n}(\mathbb{R} \times \tilde{N})$ ,  $K_0 = A_{\pi}(\mathbb{R} \times \tilde{N})$  and assume  $K = A_{\pi_0}(\tilde{N}) \subset \text{int} \tilde{N}$ . Then there exist an  $n_0 \in \mathbb{N}$  and two closed set  $N$  and  $N'$  such that for each  $n \in \{n \geq n_0\} \cup \{0\}$  there are two index pair  $\langle N_n, L_n \rangle$  and  $\langle N'_n, L'_n \rangle$  with the following properties:

- $K_n \subset N' \subset \text{int}(\mathbb{R} \times N) \subset \text{int}(\mathbb{R} \times \tilde{N})$ ,  $N'_0 = \mathbb{R} \times N'$  and  $N_0 = \mathbb{R} \times N$
- $\langle N'_n, L'_n \rangle$  (resp.  $\langle N_n, L_n \rangle$ ) is an index pair in  $N'_0$  (resp. in  $N_0$ ) w.r.t  $\pi_n$  (resp. w.r.t  $\pi = \tau \times \pi_0$  if  $n = 0$ )
- $N'_n \subset N'_0 \subset N_n \subset N_0$  and  $L'_n \subset L'_0 \subset L_n \subset L_0$ .

*Proof.* Since the proof will follow the original one almost completely and we are only interested in attractors, we only show a proof in case  $\tilde{N}$  is positive  $\pi_0$ -invariant. In particular, we will assume that  $\tilde{N}$  is an isolating block with empty exit set defined via the function  $g^-$ . For  $a > 0$  define

$$V(a) = \{x \in \tilde{U} \mid g^-(x) < a\}.$$

Then there is an  $a_0 > 0$  such that  $N := \text{cl} V(a_0) \subset \tilde{U}$ . And similar to [Ryb87, I-4.5] we can show that for  $0 < \epsilon \leq a_0$  and all  $n \geq n_0(\epsilon)$

$$K_n \subset \mathbb{R} \times V(\epsilon).$$

Define

$$N_n(\epsilon) = (\mathbb{R} \times N) \cap \text{cl}\{\tilde{y} \mid \exists \tilde{x} \in \mathbb{R} \times V(\epsilon), t \geq 0 \text{ s.t. } \tilde{x}\pi_n[0, t] \subset \mathbb{R} \times \tilde{U} \text{ and } \tilde{x}\pi_n t = \tilde{y}\}.$$

Following the proof of [Ryb87, I-12.5] we can show that  $N_n(\epsilon)$  satisfies the following properties for  $n \geq n_0(\epsilon)$

- $x \in N_n(\epsilon)$  and  $x\pi_n[0, t] \subset \mathbb{R} \times N$  implies  $x\pi_n t \in N_n(\epsilon)$

- $K_n \subset \mathbb{R} \times V(\epsilon) \subset N_n(\epsilon)$

We claim that for small  $\epsilon_0 > 0$  whenever  $\epsilon \leq \epsilon_0$  and  $n \geq n_0(\epsilon)$  then  $N_n(\epsilon)$  is positive  $\pi_n$ -invariant. If this is not true then there is a sequence  $\epsilon_m \rightarrow 0$  and

$$y_m = (s_m, z_m) \in N_{n_m}(\epsilon_m) \cap (\mathbb{R} \times \partial N).$$

By definition of  $N_{n_m}(\epsilon_m)$  there is a sequence  $\tilde{y}_m \in X$ ,  $x_m \in \mathbb{R} \times V(\epsilon_m)$  and  $t_m \geq 0$  such that  $d(y_m, \tilde{y}_m) < 2^{-m}$ ,  $x_m \pi_{n_m}[0, t_m] \subset \tilde{U}$  and  $\tilde{y}_m = x_m \pi_{n_m} t_m$ . Because  $g^-(x_m) \rightarrow 0$  and  $A_f^-(B) = A_f(B)$  we can assume w.l.o.g. that  $x_m \rightarrow x_0 \in A_f(B)$ . Admissibility and  $\pi_{n_m} \xrightarrow{\text{ssing}} \pi_0$  imply the sequence  $\{P_2(x_m \pi_{n_m} t_m)\}_{m \in \mathbb{N}}$  has a convergent subsequence and w.l.o.g.  $P_2(\tilde{y}_m) = P_2(x_m \pi_{n_m} t_m) \rightarrow z_0 \in A_f^-(\tilde{N}) = A_f(\tilde{N}) \subset \text{int } N$  (see proof of lemma 8) and thus  $P_2(y_m) \rightarrow z_0$ . Since  $P_2(z_m) \in \partial N$  we must have  $y_0 \in \partial N$ , but this is a contradiction since  $\partial N \cap A_{\pi_0}(\tilde{N}) = \emptyset$ .

Set  $N' = \text{cl } V(\epsilon)$  which is also positive  $\pi_0$ -invariant. Applying the arguments above for  $N'$  we get another positive  $\pi_n$ -invariant  $N'_n \subset \mathbb{R} \times N'$ . So we have

$$N'_n \subset \mathbb{R} \times N' \subset N_n \subset \mathbb{R} \times N.$$

Positive invariance and  $K_n \subset \mathbb{R} \times V(\delta) \subset N'_n \subset N_n$  for some  $\delta > 0$  implies that  $N_n$  and  $N'_n$  are index pairs.  $\square$

**Corollary 15.** *If, in addition to the assumption of the theorem, we assume that each  $\pi_n$  is skew product flows and  $\tilde{N}$  is a stable isolating neighborhood for  $\pi_0$  and  $N_0 = \mathbb{R} \times N$  from the theorem is strongly  $\pi_n$ -semi-admissible then  $K_n \neq \emptyset$  and each  $t$ -slice  $K_n \cap \{t\} \times X$  is non-empty and compact. Furthermore, there is an  $n_0 = n_0(\tilde{N}, \pi_0)$  and a  $\delta = \delta(\tilde{N}, \pi_0) > 0$  such that for all  $n \geq n_0$*

$$U_\delta(K_n) \subset N_n,$$

*i.e.  $K_n$  is a past attractor in the sense of Rasmussen [Ras07]. And for the sequence  $(K_n)_{n \in \mathbb{N}}$  we have*

$$\lim_{n \rightarrow \infty} \sup_{y \in K_n} \inf_{x \in K} d(P_2 y, x) = 0,$$

*i.e.  $K_n$  is upper-semicontinuous “at  $K_0$ ” as  $n \rightarrow \infty$ .*

*Remark.* In [CL03] an upper-semicontinuity is proved by assuming that a global attractor exists. Our result shows that the same is true for local attractors. We can prove without further assumptions on the system that such local attractors always exist if the perturbation is “uniformly small”. If  $K$  is the global attractor of  $\pi_0$  and the assumption (h2) of [CL03] holds then  $K_n$  must be the global attractor for large  $n$ .

*Proof.* If  $\tilde{N}$  is stable then w.l.o.g. we can replace  $\tilde{N}$  by a stable isolating block defined via the function  $g^-$  (see below). The previous theorem implies that  $N_n$  is a stable isolating neighborhood for  $\pi_n$  and  $n \geq n_0$ , i.e.  $N_n = A_{\pi_n}^+(N_n)$ . Because  $N'_0 = \mathbb{R} \times N' \subset N_n$  we have

$$N_n \cap \mathbb{R} \times X \neq \emptyset.$$

Thus there is a sequence  $s_k \rightarrow -\infty$  and  $(x_k)_{k \in \mathbb{N}}$  such that  $(s_k, x_k) \in N_n$ . Because  $N_n$  is positive  $\pi_n$ -invariant and  $N_0$  is strongly  $\pi_n$ -semi-admissible the sequence

$$(s_k, x_k) \pi_n(-s_k) \in N_n \cap \{0\} \times X$$

has a convergent subsequence and each limit point is in  $A_n^-(N_n)$  and thus in  $A_{\pi_n}(N_n) \subset \text{int } N_n'$ . By the same argument we can show that each  $t$ -slice of  $N_n$  contains a non-empty compact  $t$ -slice of the invariant set.

The upper-semicontinuity is a standard result from the index continuation (see e.g. [CR02, Corollary 4.11]).

It remains to show that there is a  $\delta > 0$  such that for large  $n$

$$U_\delta(K_n) \subset N_n.$$

Recall the definition of the function  $g^- : \tilde{N} \rightarrow \mathbb{R}^{\geq 0}$  (adjusted to the stable case)

$$g^-(x) := \sup\{\alpha(t)F(x\pi t) \mid 0 \leq t < \infty\}$$

with

$$F(x) := \min\{1, \text{dist}(x, K)\}$$

and some strictly increasing  $C^\infty$ -diffeomorphism  $\alpha : [0, \infty) \rightarrow [1, 2)$ .  $g^-$  is continuous on  $\tilde{N}$  and strictly decreasing along  $\pi_0$  outside of  $K$ . For sufficiently small  $0 < \epsilon < \frac{1}{2}$

$$B_\epsilon := (g^-)^{-1}([0, \epsilon]) \subset \text{int } \tilde{N}$$

defines a stable isolating block and  $B_\delta \subset B_\epsilon$  for  $\delta < \epsilon$ . Since  $g^-(x) < 1$  for  $x \in B$  we have  $\text{dist}(x, K) \leq g^-(x)$ . Because  $K$  is compact there exists some  $2\delta < \epsilon$  such that for all  $x \in \partial B$

$$2\delta < d(x, K) \leq \epsilon.$$

This implies

$$U_\delta(B_\delta) \subset B_\epsilon.$$

The same applies for the suspension  $\tilde{g}^- : \mathbb{R} \times \tilde{N} \rightarrow \mathbb{R}^{\geq 0}$ . Our proof of the previous theorem constructs the sets  $N_0'$  and  $N_0$  from the function  $g^-$  (resp.  $\tilde{g}^-$  in our adapted version), i.e.

$$\begin{aligned} N_0' &= (\tilde{g}^-)^{-1}([0, \epsilon]) \\ &= \mathbb{R} \times (g^-)^{-1}([0, \epsilon]) \\ &= \mathbb{R} \times B_\epsilon \end{aligned}$$

for some  $\epsilon > 0$ . By the previous argument there is a  $\delta > 0$  such that for  $N_0'' := (\tilde{g}^-)^{-1}([0, \delta])$  we have

$$U_\delta(N_0'') \subset N_0'.$$

Similar arguments as used in the proof above show that there is a stable isolating neighborhood  $N_n''$  of  $K_n = A_{\pi_n}(\mathbb{R} \times \tilde{N})$  with

$$N_n'' \subset N_0'' \subset N_n' \subset N_0' \subset N_n \subset N_0.$$

Now the inclusion sequence implies

$$U_\delta(K_n) \subset U_\delta(N_n'') \subset N_n.$$

□

**3.1. Retarded functional differential equations.** Let  $C = C([-r, 0], \mathbb{R}^m)$  be the space of continuous functions equipped with the sup-norm,  $r \geq 0$  and  $\Omega \subset C$  be an open set. For a continuous map  $x : [-r + t, t] \rightarrow \mathbb{R}^m$  and  $t \in \mathbb{R}$  we write  $x_t$  as the element in  $C$  such that  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-r, 0]$ .

If  $f_0 : \Omega \rightarrow \mathbb{R}^m$  is Lipschitz continuous then the following (autonomous) retarded functional differential equation (RFDE)

$$\dot{x}(t) = f_0(x_t)$$

induces a semiflow  $\pi_0$  such that each bounded and closed  $N \subset \Omega$  for which  $f_0(N)$  is bounded is strongly  $\pi_0$  admissible (see [HV93, Ryb87]).

*Remark.* An RFDE with  $r = 0$  is an ODE on  $\Omega \subset \mathbb{R}^m$ .

Similarly if  $f_n : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^m$  is Lipschitz continuous then the non-autonomous RFDEs

$$\dot{x}(t) = f_n(t, x_t)$$

induce (skew product) semiflows  $\pi_n$  on  $\mathbb{R} \times \Omega$ . Furthermore,  $\pi_n$  does not blow up in  $\mathbb{R} \times N$  if  $N \subset \Omega$  is closed and bounded and  $f_n(t, N)$  is bounded uniformly in  $t$ . One can even show that  $\mathbb{R} \times N$  is strongly  $\pi_n$ -semi-admissible (see proof of [Ryb87, 4.2]).

If we assume that

$$\sup_{t \in \mathbb{R}} \|f_n(t, \cdot) - f_0\| < \epsilon_n \rightarrow 0$$

then we can easily show that  $\pi_n \xrightarrow{\text{ssing}} \pi_0$ .  $\mathbb{R} \times N$  is strongly  $\{\pi_n\}$ -semi-singular-admissible.

**3.2. Semilinear parabolic equations.** The idea for semilinear parabolic equations is very similar. Suppose  $A$  is a sectorial operator and  $f_0 : X^\alpha \rightarrow X$  and  $\Omega$  are “nice” then

$$\begin{aligned} u_t &= Au + f_0(u) \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

induces a local semiflow  $\pi_0$  on some  $X^\alpha$ . Furthermore, bounded closed set  $N \subset X^\alpha$  with  $f_0(N)$  bounded are strongly  $\pi_0$ -admissible.

As above the non-autonomous equation

$$\begin{aligned} u_t &= Au + f_n(t, u) \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

induces a skew product semiflow on  $\mathbb{R} \times X^\alpha$  and  $\mathbb{R} \times N$  is strongly  $\pi_n$ -semi-admissible for a bounded closed  $N \subset X^\alpha$  with  $f_n(t, N)$  is bounded uniformly in  $t$ . If, furthermore,

$$\sup_{t \in \mathbb{R}} \|f_n(t, \cdot) - f\| < \epsilon_n \rightarrow 0$$

then  $\pi_n \xrightarrow{\text{ssing}} \pi_0$  and  $\mathbb{R} \times N$  is strongly  $\{\pi_n\}$ -semi-singular-admissible.

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